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ABSTRACT

This document discusses the importance of acceleration vs. sophistication in mathematics programs and curricula for gifted students. The discussion proceeds through three steps. First, an attempt is made to model how various kinds of gifted students interact with three dimensions of the curriculum. These dimensions are acceleration, sophistication (depth), and enrichment (breadth). Second, a description is given of the way in which one program has tried to respond to this model. This program is the mathematics program for gifted students at the University High School, the laboratory school of the University of Illinois. Finally, an attempt is made to draw a few conclusions and practical guidelines for dealing with gifted students. (HM)

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Development Report

Number 2

Acceleration vs. Sophistication in Mathematics: An Either/Or Case?

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February 1979

SE 027 799

Acceleration vs. Sophistication in Mathematics: An Either/Or Case?

Curtis C. McKnight

Much discussion in recent years has focused on the value of acceleration in mathematics programs and curricula for gifted students. The merits of acceleration are taken as well established and beyond question in most cases. Consideration then typically moves on to how best to implement acceleration.

However, it will be realized by all experienced mathematics teachers, especially those who have taught gifted students, that there are other dimensions of mathematics subject matter such as the "breadth" of the mathematics to which the student is exposed, or the "sophistication" that he is expected to display in dealing with that mathematics content. Again, most experienced teachers will feel that there must often be a trade-off between acceleration of content covered and these other dimensions of breadth and sophistication. If such a trade-off is inescapable, surely this is a factor which must be considered in decisions about implementing acceleration.

The discussion that follows is an attempt to explore this possible trade-off, whether it is inescapable, and, if not, the practical ways in which it can be minimized. The discussion draws on two areas of experience: (1) the practical experience of running a mathematics program for gifted students at University High School in Urbana; and (2) three years of federally funded research into the cognitive processes, the "human information processing," used by students in learning and doing mathematics. The

discussion proceeds through the following steps: (1) An attempt is made to model how various kinds of students interact with these various dimensions of the curriculum; (2) A description is given of the way in which one program has tried to respond to this model; and (3) An attempt is made to draw a few conclusions and practical guidelines for dealing with this question.

SOME FIRST APPROXIMATION MODELING

The modeling that follows is in some ways very "naive" and simplified. It is at best a "first approximation". However, it is modeling that is true to our experience with our own program and to our research in students' cognitive processes. First, a simple model of curriculum is offered, then a rough typology of students, and, finally, a model of how the two previous models interact.

1. Some Dimensions of Curriculum

Acceleration is one dimension, one method, of modifying and "expanding" curriculum in order to make it better serve the needs of gifted students. Figure 1 suggests two ways of representing acceleration graphically. We choose the second representation as making more clear the fact that acceleration represents an expansion of content.

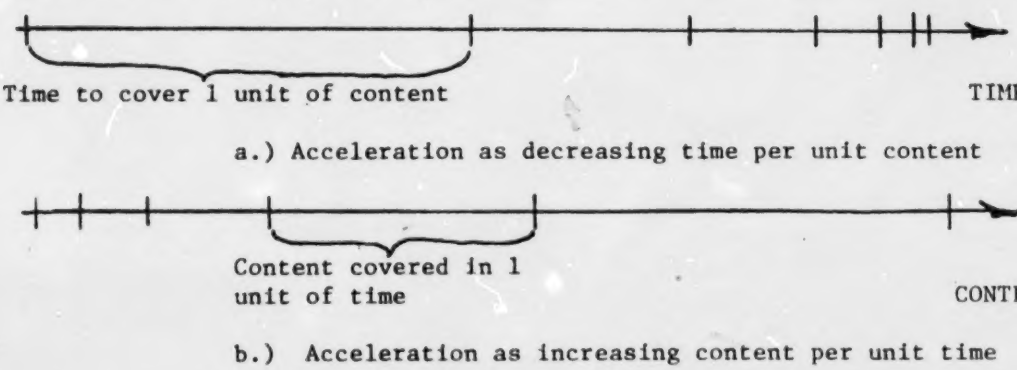

Time to cover 1 unit of content
TIME
a.) Acceleration as decreasing time per unit content
Content covered in 1 unit of time
CONTENT
b.) Acceleration as increasing content per unit time

Figure 1. Representations of Acceleration

In our program acceleration is represented by such things as: (1) covering a full-year Euclidean geometry text in one semester to allow other geometries to be studied, and (2) covering a version of the traditional high school program in three years to allow two years to cover the typical three-semester University calculus course. (Our program compresses grades 7 through 12 into five years, a significant acceleration in itself.)

Acceleration, however, is not the only dimension (method) of expanding curriculum or content. Other dimensions are enrichment (breadth) and sophistication (depth). Figure 2 shows the resulting representation of the expansion of content.

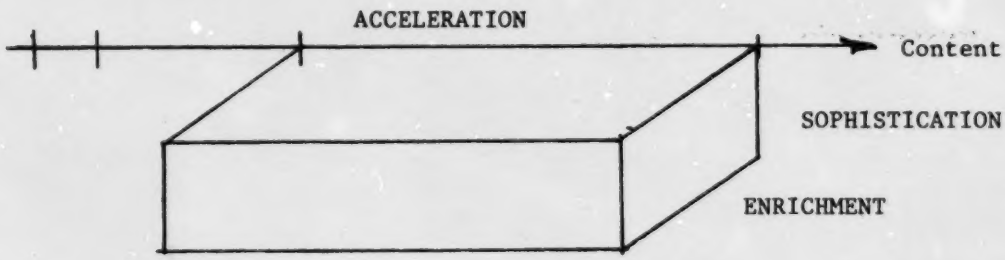


Figure 2. Dimensions of Expanding Curriculum Content

Enrichment is as familiar a concept as acceleration. It implies the introduction of topics in mathematics beyond the mainstream of the college preparatory curriculum in order to demonstrate the breadth of mathematics, increase interest and motivation, etc. In our program it is represented by such things as (1) introducing finite and projective geometries as well as Euclidean, analytic and vector geometries, and (2) introducing a course covering probability and statistics (via Monte Carlo methods),

Markov processes, etc.

Sophistication is a harder concept to define. It implies demands and encouragement for greater depth and power in the kinds of information processing used in dealing with any topic of mathematics. The following is a (not exhaustive) sample of the kinds of "sophistication" demanded and encouraged in our program.

1) Discovery of Significant Results.-Discovery problems guided only in the broadest sense and leading to significant mathematical results (e.g. de Moivre's theorem).

2) Generalization. - Encouragement to state and prove general cases (e.g., to develop a mini-theory of finite difference for quadratics, cubics, etc.)

3) Verification. - Demanding verification and proof not just in geometry but in many areas. For example, inference schemes and proofs are first introduced in first year algebra for the derivation of basic algebraic properties.

4) Meta-discussions. - Demanding not only the ability to execute a problem-solving procedure (e.g. for "falling body" problems) but also the ability to discuss the procedure in an appropriate conceptual meta-language that can lead to modifications of the procedure for greater efficiency, to handle new types of problems, etc.

5) Problem-solving. - Demanding that the student attempt to solve significant, non-routine problems.

All of these kinds of "processing" as well as many others represent demands that gifted students not only handle significant content, but also that they do so in a sophisticated fashion.

If you look again at the expansion of content represented in Figure 2, you can see that utilization of all three dimensions leads to expansion of a "volume". Thus, if acceleration, enrichment and sophistication in any one area were all doubled that would lead to an eightfold expansion of content.

The way in which such an expansion "explodes" helps us to see why, in programs with limited time resources, there must be trade-offs between these three methods of curriculum expansion.

In our program a higher priority is placed on both acceleration and sophistication. Enrichment is treated as "dependent enrichment," enrichment as a dependent variable manipulated by means of changes in the other two. For example, finite and projective geometries have been introduced into one of our geometry courses and represent significant enrichment of the curriculum. However, they were introduced solely to allow exploration of the axiomatic method of proof and verification in a context in which (1) all results did not seem "common sense" and trivial, and (2) intuition based on experience could be minimized to enhance development of other sorts of thinking. Thus the topics represent dependent enrichment introduced by manipulating sophistication. Since enrichment is so handled in our program, the rest of this discussion deals primarily with acceleration and sophistication.

2. A Typology of Students

Our experience has led us to feel that term "gifted" applied to mathematics students covers a variety of types of students. I would like to suggest a rough typology of gifted mathematics students in terms of three adjectives to be explained more fully below. They are: (1) the thorough, (2) the quick, and (3) the bright.

Two general remarks should be made before discussing this typology further. First, these types are at most partly independent, semi-orthogonal. Both the quick and the bright will hopefully be thorough as well, but that is not always the case and there are certainly thorough students who show no signs

of quickness, etc. It will be seen that many students will be what I call quick but not what I call bright and vice versa. Of course, the best of all categories is those students who are all three. Second, these adjectives are both (1) labels in an intuitive typology based on experience and (2) labels for emerging clusters of related cognitive processing skills which tend to validate the intuitive typology. Thus they relate both to our practical program experience and to our research findings.

The first two types are relatively easy to describe and, I think, fit well with the experience of most mathematics teachers. The thorough represents the kind of student who is careful to do all assigned work and is careful to master all content presented. In addition, this type of student often takes responsibility for monitoring and evaluating his own preparation, and for "learning the material" instead of just "doing what's assigned." This characteristic is a foundation needed for either of the other two characteristics to lead to fruitful experiences and growth in mathematics learning.

The quick represents the kind of student who is able to grasp quickly topics presented, to follow (but not necessarily to generate) arguments easily, to develop easily his own procedures for solving routine kinds of problems to which he has been exposed. The natural habitat of such a student is a curriculum expanded by acceleration. Quickness implies an ability to deal well with acceleration.

It is, of course, hard to give an illustration of quickness by static samples of student work, but three brief examples may help somewhat. These are all with ninth grade students. Figure 3 is typical student derivation

(re-typed) of an algebraic property. It represents the kind of work done by a quick ninth-grader after about 3 to 4 days of working with derivations. Figure 4 represents a student answer on an hour exam to a question about the delta process. It is only one of several questions on the exam and came after less than two weeks of work on slopes, derivatives, maxima and minima, etc. Figure 5 is a student final exam answer when asked to derive the quadratic formula. Again, this was one of several questions in one hour, and it was on material which the student had not seen for months except for a brief review a few days before. These kinds of questions in this kind of time frame represent a demand for the ability I have been calling quickness, an ability needed tc respond effectively to an accelerated curriculum.

The bright is a category somewhat more difficult to describe than the previous two. In our research we are coming to identify and document clusters of cognitive processing skills that seem closely associated with those students we would characterize informally as bright rather than quick. In general, nontechnical language, the following are some of those skills:

1.) They are able to generalize spontaneously and fluently.

2.) They are often insightful and intuitive.

3.) They possess good general problem-tackling strategies and a good heuristic repetoire.

4.) They tend to cross reference the things they learn with what they already know in a way that often allows them to produce a useful piece of knowledge in a context far different from the one in which they learned it.

Some examples of these things are in order. Again all of the examples

Theorem: $A + A = 2 \times A$

Proof:

1)	$\square = \square$	Reflexive property of equality
2)	$2 \times A = 2 \times A$	U.V.[1], $2 \times A \rightarrow \square$, in line 1.
3)	$1 + 1 = 2$	Definition of 2.
4)	$(1 + 1) \times A = 2 \times A$	P.N.[2], from line 2, using line 3.
5)	$\square \times \triangle = \triangle \times \square$	Commutative Law for Multiplication.
6)	$(1 + 1) \times A = A \times (1 + 1)$	U.V. $1 + 1 \rightarrow \square$, $A \rightarrow \triangle$, in line 5.
7)	$A \times (1 + 1) = 2 \times A$	P.N., from line 4, using line 6.
8)	$\square \times (\triangle + \triangledown) = (\square \times \triangle) + (\square \times \triangledown)$	Distributive Law
9)	$A \times (1 + 1) = (A \times 1) + (A \times 1)$	U.V., $A \rightarrow \square$, $1 \rightarrow \triangle$, $1 \rightarrow \triangledown$, in line 8
10)	$(A \times 1) + (A \times 1) = 2 \times A$	P.N., from line 7, using line 9.
11)	$\square \times 1 = \square$	Law for 1
12)	$A \times 1 = A$	U.V., $A \rightarrow \square$ in line 11.
13)	$A + A = 2 \times A$	P.N., from line 10, using line 12.

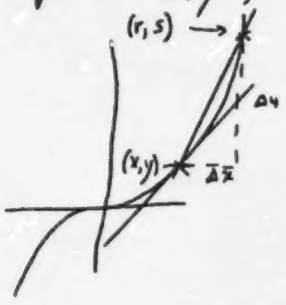
Q.E.D.

Figure 3. A Student Derivation [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1, Summer, 1976, pp. 110-111]

10e Toshi N.

Grade 7 April 9, 1976

Please use the "delta process" to find the slope y', at the point (x, y), of the curve $y = x^3$.



$$(\text{slope})_{\text{secant}} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{s-y}{r-x} = \frac{r^3-x^3}{r-x}$$

$$r-x \;\overline{)\; r^3 \qquad -x^3}$$

$$R^2 + xR + x^2$$

$$r^3 - xr^2$$

$$xr^2 - x^2r$$

$$x^2r - x^3$$

$$x^2r - x^3$$

$$0$$

$$\frac{(r-x)(r^2+xr+x^2)}{r-x} = r^2+xr+x^3$$

$$y' = (\text{slope})_{\text{curve}} = (\text{slope})_{\text{tangent}} = \lim_{(r,s)\to(x,y)} \left[(\text{slope})_{\text{secant}}\right]$$

$$= \lim_{(r,s)\to(x,y)} \left[r^2+xr+x^2\right]$$

$$= x^2+x\cdot x+x^2 = 3x^2$$

Figure 4. Test Answer on Delta Process [Reproduced from Journal of Children's Mathematical Behavior, Supp. No., 1, Summer, 1976, pp. 126-127].

Final Exam - Gr. 9 May 7, 1976
Pt. 1 (1 hour)

1.

Please write out a careful derivation of the quadratic formula,

ax^2+bx+c

100

$x^2+\frac{b}{a}x = \frac{-c}{a}$

$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$

$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

$\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

$\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0$

$\left(x+\frac{b}{2a}\right)^2 - \frac{\sqrt{b^2-4ac}}{2a} = 0$

$\left[\left(x+\frac{b}{2a}\right) - \frac{\sqrt{b^2-4ac}}{2a}\right]\left[\left(x+\frac{b}{2a}\right) + \frac{\sqrt{b^2-4ac}}{2a}\right] = 0$

$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ $\qquad$ $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Q.E.D.

Figure 5. Final Exam Answer. Deriving the Quadratic Formula. [Reproduced from Journal of Children's Mathematical Behavior Supp. No. 1, Summer, 1976, p. 182]

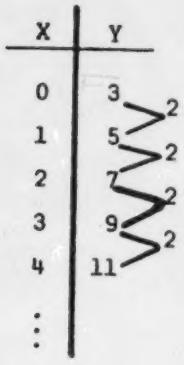
except one will be taken from the work of ninth grade algebra students.

Figure 6 represents some student generalizing with finite differences. Sections a and b are the kind of work which had been going on in class, looking at tables of finite differences to "fit" a given table of "input-output" pairs to linear, quadratic, and cubic equations. One student, a 12-year-old girl, towards the end of a period as other students worked on homework, asked if she could try to solve the cubic case in general. Figure 6c is the work she produced after about 20 minutes. The cubic case is stated in general and a convincing demonstration offered of the way it works. The circled portions were done by the student to indicate how she could find A,B, and D, and she stated that she could then easily find C given one pair other than $x = 0$.

Figures 7 and 8a-e relate to work (an example of which was shown earlier) using limits intuitively to show $f'(x) = 3x^2$ when $f(x) = x^3$, etc. At the end of this work, two students decided to tackle the general case. Figure 7 is the work of a 12-year-old boy who poses the case for $f(x) = x^n$ easily and carries out the solution in a workman-like fashion. Figure 8 is the work of the girl mentioned earlier and shows a related but different phenomenon. She does the same general case, but carefully places it in the conceptual framework by which the material was explained. She is here working not only to generalize a specific aspect of the problem but also to abstract and consolidate a general conceptual framework for the whole type of problem.

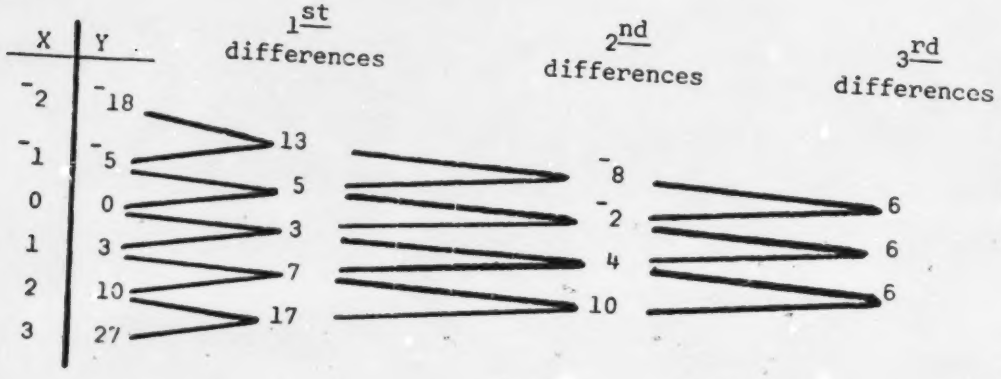
Figure 9a is an example of student insight and intuition. This is a subtraction algorithm spontaneously developed by a third grade boy named Kye. The teacher had started a standard discussion about not being able to take 8 from 4, in order to introduce the needed modification into the subtraction algorithm. Kye replied that you could take 8 from 4 (they had done some work with negative integers earlier) and proceeded to demonstrate his algorithm. The teacher, although surprised, went along with him and asked the class if he was correct, leading them all to explore his algorithm.

(a)



$2x + 3 = y$

(b)



$6 \div 3! = 6 \div 6 = 1$ $\quad y = x^3 + \cdots$

(c)

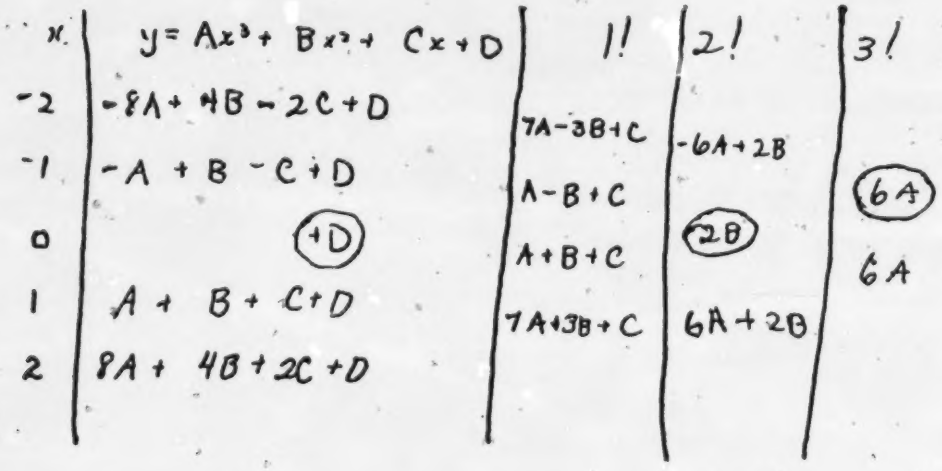


Figure 6. Generalizing with Finite Differences. [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1, Summer, 1976, pp. 143-144.]

General Equation for LIMIT Joshua

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{s-y}{R-x} = \frac{R^n - x^n}{R-x}$$

$$R-x \;\overline{)\; R^n - x^n \;\big|\; R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)}}$$

$$R^n - xR^{n-1}$$

$$xR^{n-1}$$

$$xR^{n-1} - x^2R^{n-2}$$

$$x^2R^{n-2}$$

$$x^2R^{n-2} - x^3R^{n-3}$$

$$= \frac{(R-x)(R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)})}{R-x}$$

$$= (R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)})$$

$$\lim \frac{\Delta y}{\Delta x} = \lim_{R-x \to 0}$$

$$(R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)}) = \underbrace{x^{n-1} + x^{n-1} + x^{n-1} \ldots\ldots x^{n-1} + x^{n-1}}_{n \text{ times}} = nx^{n-1}$$

Figure 7. Generalizing with Limits. I. [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1, p. 130].

General Equation for LIMIT

Joshua

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{s-y}{R-x} = \frac{R^n - x^n}{R-x}$$

$$R-x \;\overline{)\; R^n - x^n \;\big|\; R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)}}$$

$$R^n - xR^{n-1}$$

$$xR^{n-1}$$

$$xR^{n-1} - x^2R^{n-2}$$

$$x^2R^{n-3}$$

$$x^2R^{n-3} - x^3R^{n-3}$$

$$= \frac{(R-x)(R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)})}{R-x}$$

$$= (R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)})$$

$$\lim \frac{\Delta y}{\Delta x} = \lim_{R-x \to 0}$$

$$(R^{n-1} + xR^{n-2} + x^2R^{n-3} \ldots\ldots + x^{n-1}R^{n-(n-1)}) = \underbrace{x^{n-1} + x^{n-1} + x^{n-1} \ldots\ldots x^{n-1} + x^{n-1}}_{n \text{ times}} = nx^{n-1}$$

Figure 7. Generalizing with Limits. I. [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1, p. 130].

Kazuko
4/7/76

Math Proof: $(\text{slope})_{x^n} = nx^{n-1}$

To find the slope of any point on the curve:

$y = x^n$

we must find the slope of the tangent line at that point. However, we don't know very much about the slope of curved lines. Using Newton's idea of turning a new problem into an old one, we try to find the slope of a straight line, the tangent line.

A portion of the curve

$y = x^n$

might look like this:

First, we take the point where we want to find the slope of that portion of the curve, and give it a name:

From here, we have to take another point as close to (x,y) as possible to find the secant line.

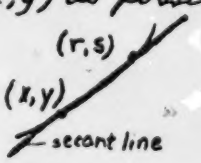
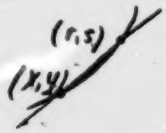


Figure 8a-e. Generalizing with Limits. II. [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1, pp. 131-135].

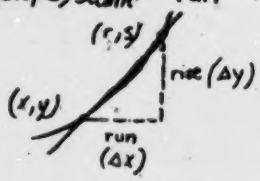
As the two points come closer and closer, the slope of the secant comes closer and closer to the slope of the tangent.

Now, we name the second point, forming a secant.



The slope of this line, the secant, is easy to find, using what we know about the slope of straight lines.

$(\text{slope})_{\text{secant}} = \frac{\text{rise}}{\text{run}}$ when:



Since the y-coordinate of (r,s) is s, and the y-coordinate of (x,y) is y, rise equals s-y because

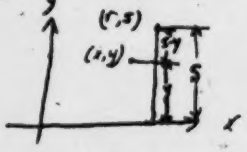
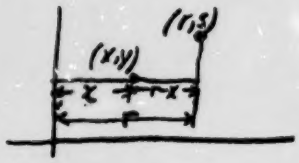


Figure 8b

Similarly, run equals $r-x$ because:



So, $(\text{slope})_{\text{secant}} = \frac{\text{rise}}{\text{run}} = \frac{s-y}{r-x}$.

Now, since $y = x^n$, we can write x^n instead of y. And because (r,s) is a point on the curve $y = x^n$, r must bear the same relationship to s. Therefore, $s = r^n$, and we can write r^n instead of s.

So, $(\text{slope})_{\text{secant}} = \frac{\text{rise}}{\text{run}} = \frac{s-y}{r-x} = \frac{r^n - x^n}{r-x}$

But now, although seem to have solved the problem, a difficulty arises. If we move (r,s) so close to (x,y) so that $r \to x$, $s \to y$, then put in the equation above, we get

$$\frac{r^n - x^n}{r-x} = \frac{x^n - x^n}{x-x} = \frac{0}{0},$$

which gives us an undefined value. However, we can get out of this difficulty by not using the

Figure 8c

fraction form; in other words, if we divide the rise by the run, we can get rid of the fraction.

$$
\begin{array}{r}
r^{n-1}+r^{n-2}x \ldots rx^{n-2}+x^{n-1} \\
r-x \;\overline{)\; r^n \ldots \qquad -x^n} \\
\underline{r^n - r^{n-1}x} \\
r^{n-1}x \\
\underline{r^{n-1}x - r^{n-2}x^2} \\
\vdots \\
\underline{-r^2x^{n-2}} \\
r^2x^{n-2} \\
\underline{r^2x^{n-2} - rx^{n-1}} \\
rx^{n-1} - x^n \\
\underline{rx^{n-1} - x^n} \\
0
\end{array}
$$

Therfore, the slope formula would be:

$$(\text{Slope})_{\text{secant}} = r^{n-1}+r^{n-2}x+r^{n-3}x^2 \ldots +r^2x^{n-3}+rx^{n-2}+x^{n-1}$$

From this equation, we can tell that there are n terms in it since all the terms but one have an x in them and there are n-1 terms with x in them since the powers of x increase by one (n-1) and there is one more term without an x (+1). Therefore, there are n terms in that equation.

Now, if (r,s) gets closer and closer to (x,y) and $r \to x$, we get:

$$(\text{slope})_{\text{tangent}} = x^{n-1}+x^{n-2}x+x^{n-3}x^2 \ldots +x^2x^{n-3}+xx^{n-2}+x^{n-1}$$
$$= x^{n-1}+x^{n-1}+x^{n-1} \ldots +x^{n-1}+x^{n-1}+x^{n-1}$$

Figure 8d

And, because there are n terms in the equation

$(\text{slope})_{\text{tangent}} = nx^{n-1}$

And if the curve was $y = kx^n$ then we would proceede as before but the equations would have to be multiplied by k. Therefore, the final result would be:

$(\text{slope})_{\text{tangent}} = knx^{n-1}$

Figure 8e

$$\begin{array}{r} 64 \\ -28 \\ \hline \end{array}$$

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \end{array}$$

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \\ 40 \end{array}$$

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \\ 40 \\ \hline 36 \end{array}$$

(a)

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]^2 = i$$

$i^{(\frac{1}{N})}$ is on the unit circle with an angle (90/N) degrees. [Michael]

"That fits with $i^0 = 1$." [Philip]

(b)

Figure 9. (a) Kye's Subtraction [JCMB, Supp. No. 1, pp. 72-73]. (b) Philip's Cross-Referencing [JCMB, Vol. 2, No. 1, Spring, 1978, pp. 133-134].

$$\begin{array}{r} 64 \\ -28 \end{array}$$

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \end{array}$$

$$\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]^2 = i$$

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \\ 40 \end{array}$$

$i\left(\frac{1}{N}\right)$ is on the unit circle with an angle (90/N) degrees. [Michael]

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \\ 40 \\ \hline 36 \end{array}$$

"That fits with $i^0 = 1$." [Philip]

(a) (b)

Figure 9. (a) Kye's Subtraction [JCMB, Supp. No. 1, pp. 72-73]. (b) Philip's Cross-Referencing [JCMB, Vol. 2, No. 1, Spring, 1978, pp. 133-134].

Theorum: $A+A=3\times A$ *is false if* $A\neq 0$

Proof:

(*supposing that the ~~theorum~~ equation is ~~false~~ true:*)

1) $A+A=3\times A$

2) $3=2+1$ D3

3) $A+A=(2+1)\times A$ P.N. *from line 1 using line 2*

4) $\square\times(\Delta+\nabla)=(\square\times\Delta)+(\square\times\nabla)$ DL

5) $\square\times\Delta=\Delta\times\square$ CLM

6) $\square\times(\Delta+\nabla)=(\Delta+\nabla)\times\square$ U.V. $\square\rightarrow\square$, $\Delta+\nabla\rightarrow\Delta$ *in line 5*

7) $(\Delta+\nabla)\times\square=(\square\times\Delta)+(\square\times\nabla)$ P.N. *from line 4 using line 6*

8) $(2+1)\times A=(A\times 2)+(A\times 1)$ U.V. $2\rightarrow\Delta$, $1\rightarrow\nabla$, $A\rightarrow\square$ *in line 7*

9) $A+A=(A\times 2)+(A\times 1)$ P.N. *from line 3 using line 8*

10) $2=1+1$ D2

11) $A+A=(A\times(1+1))+(A\times 1)$ P.N. *from line 9 using line 10*

12) $A\times(1+1)=(A\times 1)+(A\times 1)$ U.V. $A\rightarrow\square$, $1\rightarrow\Delta$, $1\rightarrow\nabla$ *in line 4*

Figure 10a-b. Derivation Using a Contradiction [Reproduced from Journal of Children's Mathematical Behavior, Supp. No. 1.]

13) $A+A=(A\times 1)+(A\times 1)+(A\times 1)$ P.N. from line 11 using line 12

14) $\square \times 1=\square$ LOM

15) $A\times 1=A$ U.V. $A\rightarrow\square$ in line 14

16) $A+A=A+A+A$ P.N. from line 13 using line 15

17) $\square=\square$ RPE

18) $(A+A)+^{\circ}(A+A)=(A+A)+^{\circ}(A+A)$ U.V. $(A+A)+^{\circ}(A+A)\rightarrow\square$ in line 17

19) $(A+A)+^{\circ}(A+A)=(A+A+A)+^{\circ}(A+A)$ P.N. from line 18 using line 16

20) $(A+A)+^{\circ}(A+A)=[A+(A+A))+^{\circ}(A+A)$

21) $(A+A)+^{\circ}(A+A)=A+((A+A)+^{\circ}(A+A))$

22) $\square+^{\circ}\square=0$ L°

23) $(A+A)+^{\circ}(A+A)=0$ U.V. $A+A\rightarrow\square$ in line 22

24) $0=A+0$ P.N. from line 21 using line 23

25) $\square+0=\square$ LZA

26) $A+0=A$ U.V. $A\rightarrow\square$ in line 25

27) $0=A$ P.N. from line 24 using line 26

28) if $A=1$, then $0=1$ U.V. $1\rightarrow A$ in line 27

29) $0\neq 1$

30) lines 28 and 29 are in contradiction; therefore $A+A=3\times A$ is false and the theorem is true

Figure 10b

No one had discussed the structure of the subtraction algorithm for Kye but he showed some insight into it. No one cued him to look for a novel algorithm. He started from one fact ("you can take 8 from 4") and assumed that you could work out the algorithm using this, a clear example of some sort of intuition.

Figure 9b is the work of two ninth-grade boys. The class had developed the matrix representation of i and then the standard algebra of complex numbers, their representation on the Argand plane and their polar form. On the day represented here they had worked on i and what its polar from would be. This led Michael to conjecture that $i^{\frac{1}{N}}$ would have an angle of 90/N. At this point Philip spontaneously offered that this "fits with $i^0 = 1$." Philip had cross-referenced this interesting conjecture with something he knew about exponents, relating 1/N to 0, and (90/N) to the position of 1 on the Argand plane.

Figure 10 is something of an example of the use of heuristics and strategy in general, again by a ninth-grade girl. This derivation involved using an inference scheme for indirect proof and finding a contradiction in order to prove something was false. What this leads to is the inability to use any routinized derivation-producing procedure. Instead the student must use more general strategies and heuristics about derivations to "solve" this derivation piece at a time. This is the work of only one of several students who solved the problem.

This then is at least an intuitive idea of what I mean by a "bright" student as opposed to a "quick" student. (Of course, several of the students in the examples were both bright and quick.) I would assert that the bright have the same kind of natural affinity for curricular sophistication that the quick have for curricular acceleration. They are at home with the demand for depth and sophistication of information processing, since it "fits" the kind

of cognitive processing skills which they demonstrate effectively.

3. A Model of Interaction

These two previous models lead to a model of their interaction. This is found in Figure 11.

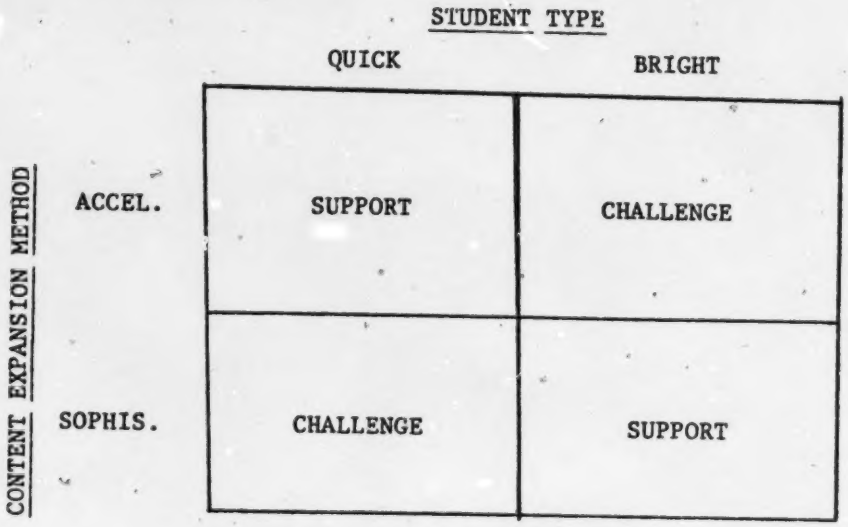


Figure 11. A Model of Interaction

This representation points out that there are four possible combinations or interactions between the content expansion methods and the typology of students, both described earlier. These four combinations, it seems to me, fall into two categories: (1) those that support and affirm the abilities of students of the given type, and (2) those that challenge students of a given type to develop and practice skills which they do not necessarily have well developed.

This model, along with three assumptions, seems to lead to an inescapable conclusion. The three assumptions are as follows:

1) Almost any program for gifted students in mathematics (no matter how small) will contain both quick students and bright students.

2) It is important that every student be supported, that he have his strengths affirmed and valued.

3) It is important that students develop a wide range of skills, if possible, and thus it is important that every student develop as much as possible those skills which are not his strengths. Thus every student should be challenged by work in the areas not his strength.

If these three assumptions are made and the model accepted, the inescapable conclusion follows:

Conclusion: A mathematics curriculum for gifted students must provide for both acceleration and sophistication.

Of course, this conclusion could be bolstered by arguments such as (1) once mathematics is learned, sophistication lasts while acceleration remains only as a "head start", the significance of which decreases as years pass; or (2) sophistication which, in a sense, is "process acceleration (or precocity)" is just as important as what is usually called acceleration but might be better called "product acceleration (or precocity)."

But why go to such lengths to make a case for sophistication as well as acceleration? I might suggest the following three reasons:

1) There are popular notions of acceleration as the panacea for curriculums for gifted mathematics students that almost say, in effect, "Plunk that gifted 12-year-old or 14-year-old down in a university calculus class, the sooner the better." While it is probably true that for such a student it is better to do this than to do nothing special, it is still true that such classes are often taught in the most routinized fashion imaginable and in ways that demand little more than thoroughness. Precocious or accelerated presentation of topics does not automatically produce a demand for mathematical sophistication.

2) Acceleration has a tremendous public relations advantage for the harried teacher or administrator who is trying to start or maintain a program for gifted mathematics students. It is effective to be able to say, "We will have these children ready to tackle calculus by the time they are sophomores or juniors." It is sometimes much harder to demonstrate, let alone "sell", sophistication to non-mathematicians, especially in this day of "back to the basics." Acceleration seems to have a natural continuity with "back to the basics," at least in comparison to sophistication which has a corresponding discontinuity.

3) As the model in Figure 2 suggests, there really is sometimes something of a trade-off between acceleration and sophistication. Thus a program that provides for both must be something of a compromise. If we are to continue to insist on such compromises in the face of the difficulty of providing for them in actual programs, we must be thoroughly convinced of their worth (and thus of the worth of the two things leading to the compromise). The case for sophistication has not had many effective advocates in recent years.

We have thus come to the point in the discussion where we are affirming both that (1) acceleration and sophistication must both be provided for; and that (2) there is a necessary trade-off between acceleration and sophistication. This leads us to ask, (1) "Can we minimize this trade-off, find ways and structures for getting more of both into our program?" and (2) "How do we go about making those necessary compromises entailed by this trade-off?" At this point I must frankly admit that my stating of the questions may be better than any answers I offer. Answering these questions seems to be a shared, communal task. We must answer them together by sharing our experiences, our insights, and our "trials and errors." It seems just here that I.O.E. gifted programs could offer the most strategic support and facilitation.

A DESCRIPTION OF ONE PROGRAM

For my initial contribution towards a communal answer, I can offer only a description of one program as it has evolved over the last several

years, and perhaps draw a few conclusions from our experience.

Let me begin by describing the setting of our program. It is set in University High School, the laboratory school of the University of Illinois. The school has about 200 students drawn from a population of about 90,000 persons. It is a school for "gifted", at least in the sense of "academically talented." The students are admitted after a series of exams (including an arithmetic achievement test) and interviews and on the basis of a composite rating. Since "supply" always exceeds "demand," these criteria are quite selective, but, since the overall source population is relatively small, the student body represents quite an ability range, both in basic abilities but especially in any one subject area such as mathematics. We thus get a fair number of students representing the quick and bright, the quick, the bright, and the "generally above average but not very good in math."

Students typically enter in their seventh grade year for a 5 year program (replacing 6 public school grades). Three math courses are required (Math 7, 9 and 10). Class sizes for the first two years are kept at 15-20 by offering more sections, although some go to 25. Classes for sophomores and juniors are about 25-30, while those for seniors are again 10-20. Almost all students take math all 5 years but upper division classes are smaller because there are a few more options.

There are 5 basic courses in the program: Math 7, 9, and 10, Finite Math and Statistics, Calculus I and Calculus II (the latter three being elective). Finite Math is a course being developed by Profs. Hirstein and Travers at the University of Illinois and includes probability, statistics, matrices, Markov processes, introduction to limits and a very concrete look at elementary functions. It features a heavy use of computers.

Calculus I and II cover Thomas' Alternate Edition calculus text and follow the syllabus for the University of Illinois three course sequence. (The fact that we spread a three semester sequence over 4 semesters reflects the fact that (1) we have 45 minute class periods and one-half hour of homework per night, and (2) we stress in some classes a sophistication greater than that demanded in any but a University Honors section.)

Probably the more interesting courses are Math 7, 9, and 10. The first two focus on algebra, trigonometry, beginning analytic geometry, matrices and a few other topics arranged according to a spiral approach. Math 10 is devoted primarily to Euclidian, analytic and vector geometry. Any of these courses represents a significant acceleration over a typical curriculum.

But what of sophistication? This begins an interesting story. Originally all sections of all courses used the same accelerated syllabus, and the same set of activities (which manipulates sophistication), and the same classroom organization structures. Classes were taught in a format focussing on exploratory learning featuring lots of discovery tasks, discussions of these explorations, the following out of tangents, etc. The student was made responsible for monitoring his own preparation, for doing enough drill to consolidate needed skills and so on. Mastery at any fixed time was de-emphasized by allowing re-takes of tests until material was mastered. It was felt that such a classroom environment would foster students' development of sophistication and that talented students and well-designed activities would still make possible coverage of an accelerated syllabus.

What emerged after some years' experience was that students tended to separate themselves into two groups based on their response to this classroom environment and these activities. Some students performed quite poorly in these circumstances without extensive individual tutoring and test re-takes. Clearly an alternative was called for.

A second type of section was developed to cover largely the same material. The classroom environment was less open: more explicitly structured tasks and activities, assigned drill, more reception learning rather than discovery, more teacher monitoring of student preparation, etc. The activities were clearly specified and geared towards meaningful reception with adequate consolidating drill. Tangents were typically not followed.

Such a change of strategy introduced inevitable syllabus changes. Much of the previous content (e.g., intuitive limits) was learned usefully only in discovery tasks and demands for sophistication. The syllabus of each course was re-examined for an essential, core syllabus which could be presented in either format and which was necessary for the next course, with other topics made optional. This became the syllabus for the more traditional, "core" sections, while the original format and syllabus were used for another kind of "extended section."

Thus, there developed a type of "tracking within tracking" program with two kinds of sections for each course. Interchange between the two kinds of sections was kept very open (always between courses and often within courses). Students were guided in finding which section suited them best but were encouraged to try the more demanding sections where reasonable. All sections of Math 7, the first course, continued to be taught in the more demanding format with the more traditional approach

offered as an option beginning with Math 9, on the basis of performance in Math 7.

Essentially the extended sections provided for both acceleration and sophistication while the more traditional sections provided primarily for acceleration only. Students that were "quick and bright" typically ended up in the extended sections while those that were primarily "quick" tended to end up in the more traditional (but still highly accelerated) sections. Those that were "bright" but not particularly "quick" or "thorough" often (but not exclusively) ended up in the extended sections, but often had difficulty with the aspects of those courses that demanded consistent, extended performances.

Further experience produced a little "fine tuning" of this system. A greater, more explicit demand for foundation skills was introduced into the extended sections. More importantly, frequent sophistication-demanding tasks were introduced back into the more traditional sections. All students seem to respond to an occasional (as opposed to constant) exposure to such tasks. Further, "quickness" and "brightness" seem to have different maturation patterns, so this element of demand for sophistication was kept even in the acceleration-oriented courses to encourage the maturing of "brightness." The frequent shifting between kinds of sections indicates that the strategy was largely successful.

SOME CONCLUSIONS

At least two practical conclusions seem reasonable based on the conceptualizations presented earlier and the experience just described.

First, the maturation patterns of "quickness" and "brightness" seem to be different. Too little is yet known about these clusters of skills to really explain this or allow any really definitive statements, but it seems clear at the level of practical experience that there is a difference. This suggests that any program decisions should not assume a clear ability to characterize students in this way, but rather should allow for the gradual emergence of the patterns and should provide for "adjustments" to take advantage of the growth of students in either area.

Second, provisions for sophistication can be introduced either via alternate courses or by a variety of tasks and activities within one course. Certainly if a gifted program is large enough to allow a variety of courses, that would be a desirable option. However, any program or course can introduce sophistication-oriented tasks within even one course. We normally resist this, thinking that such occasional ventures can hardly be fruitful, but our experience suggests otherwise. The model discussed earlier also suggests the worth of such activities.

This suggests that any program for the gifted in mathematics should (1) take a careful look at the emphases of the program with respect to acceleration vs. sophistication; (2) take a careful look at the students involved to determine something of their characteristics with respect to the typology presented earlier; (3) take a careful look at the syllabus and activities of each math course for gifted students to look for possible places in which sophistication can be introduced; (4) try to build up a repetoire of sophistication-oriented activities to draw from in supplementing acceleration; and (5) realize that such sophistication activities do demand time and patience (they require an attitude and approach which is often different than that for a "fast-track" accelerated setting; the emergence of this attitude often comes slowly) which suggests some thought should

be given to which syllabus items might be traded as less essential to provide the time needed for sophistication.

This, at least, is our current approach and the path we will be pursuing in the near future. (1) We will continue to tinker and experiment with our program to try to gain more practical insight. (2) We will continue our research to try to understand children's cognitive processing in mathematics and to more precisely characterize quickness, brightness and other clusters of skills that emerge. (3) We will continue our curriculum development efforts to enlarge our own repetoire of sophistication activities for our own use and that of others.

Much of the cognitive research that we have done is available in a more detailed, technical form in Vol. 2, No. 1, of the Journal of Children's Mathematical Behavior. Some documentation of our program is available in JCMB, Supplement No. 1. Many of our activities, unfortunately, are not in easily available form, but we invite you to write or visit us. We would be glad to offer what we can to you and even more glad to learn what we can from you.